Planar bifurcation subject to multiplicative noise: Role of symmetry

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The effect of multiplicative noise on a system described by two modes close to a bifurcation point is investigated. The bifurcation is assumed stationary and noise acts as random coupling between these modes. An analytic formula that predicts the onset of instability is derived, and the domain of existence of on-off intermittency is calculated based on an eigenvalue problem. This approach, confirmed by numerical simulations of the Langevin equations, allows quantifying the possible effects of the noise. The stability and the on-off behavior are shown to be very sensitive to deviations of the deterministic system from the case where both modes grow with equal rate and the system displays a continuous symmetry associated to rotation in phase space. In general, a noise term that breaks this continuous symmetry will increase the domain of instability of the system while a noise term that preserves the symmetry reduces the domain of instability.

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I. INTRODUCTION

In many situations patterns are formed as a result of an instability that breaks a continuous symmetry. Rayleigh-Bénard convection in an horizontal fluid layer heated from below [1], Faraday instability in a vertically shaken fluid layer [2], and Rosensweig instability of a ferrofluid plunged in a vertical magnetic field [3] are examples of instabilities that develop spatial structures of finite length scale. These structures are likely to break the originally translational and rotational invariant basic states. Besides the instability, the symmetry of these systems can also be broken by the presence of uncontrolled parameter changes or fluctuations (noise) that are typically present in any realistic experiment. Close to onset, we expect that fluctuations modify the instability process and the resulting spatial structure.

Another system that can exhibit symmetry-breaking instabilities and motivates our work is the dynamo instability that is responsible for the generation of the magnetic field in astrophysical objects. Planets and stars are at first approximation invariant under rotation around their spin axis. In some cases this symmetry is preserved by the large scale magnetic field. The Earth, for example, generates a roughly axial dipole. In other planets such as Neptune and Uranus, a transverse dipole appears to be present and the original symmetry is broken. Laboratory experiments [4–6] of the dynamo instability also display a continuous symmetry that in some cases is broken. In Riga [4], a helical flow in a cylinder generates a magnetic mode that breaks invariance under rotation around the cylinder axis. The Karlsruhe experiment [5] is composed of an array of pipes in each of which a helical flow is driven. For a large scale magnetic field, the flow properties are invariant under rotation around the pipe axis. The observed magnetic field is a transverse dipole that breaks this symmetry. In the Von Karman Sodium experiment [6], a turbulent swirling flow is generated by two counter-rotating coaxial disks. In this case the large scale magnetic field is an axial dipole that preserves the invariance under rotation around the disks axis.

Typically, realistic flows that generate the dynamo instability are highly turbulent [7]. These turbulent fluctuations are likely to play an important role, in particular in the vicinity of the onset of dynamo action. Since the equation for the magnetic field is linear, the fluctuations act multiplicatively on the unstable modes and can be modeled as multiplicative noise. Instabilities in the presence of multiplicative noise have been studied when one mode is close to criticality. It has been shown that the onset of instability can be shifted compared to the deterministic (noise free) case [8]. Slightly above onset, on-off intermittency takes place where the noise alternatively drives the system into its large amplitude phase (on phase) and close to zero (off phase) [9–14]. This behavior is exemplified by the time series shown in Fig. 1. On-off intermittency has been demonstrated in numerical simulations of the magnetohydrodynamic (MHD) equation [15–17]. However no on-off regime has been found in experimental dynamos, probably because the turbulent fluctuations have too small intensity at small frequencies [18].

Most theoretical investigations of the effect of multiplicative noise have been restricted to single mode bifurcations. In this article, we study how multiplicative noise affects the process of instability when the system displays a continuous symmetry or when this symmetry is broken. This is done by considering the generic case of two modes close to a stationary bifurcation.

We derive the onset of instability and determine the regime of parameters on which on-off intermittency exists. Section II describes in some detail the system under study. In

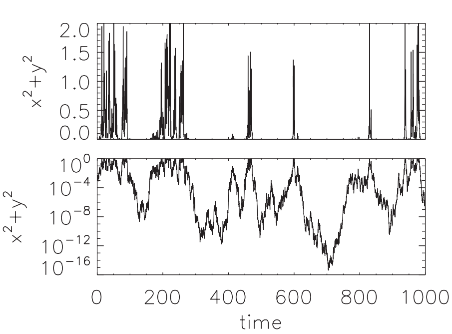

FIG. 1. A typical signal of a system that exhibits on-off intermittency, in linear (top panel) and log-linear scale (bottom panel).

Sec. III we derive a general criterion for the instability. Our results for specific examples are shown in Secs. IV and V and we draw our conclusions in the last section.

II. GENERIC PLANAR INSTABILITY WITH A CONTINUOUS SYMMETRY

We consider two modes of amplitude $x(t)$ and $y(t)$ for which $x=y=0$ is a stationary solution that can be unstable. Close to the onset of instability the evolution equation for the amplitude $A=x+iy$ can be written at linear order as

$$\dot{A} = pA + q\bar{A}. \tag{1}$$

The parameters p and q are set real so that whatever their values only stationary instabilities can occur. When $q=0$, this equation is phase invariant, i.e., is invariant under the transformation $A \to Ae^{i\theta}$. This is the continuous symmetry that can be broken by the instability. Going back to the examples related to pattern forming instabilities and dynamo effect, this symmetry traces back to the translation or rotation of the pattern or to the rotation of the transverse dipolar mode. When $q \neq 0$, the system does not display any symmetry because the two modes have different growth rates say $\mu=p+q$ and $\sigma=p-q$. By varying q, we can change the system from a situation where a continuous symmetry exists to a situation where it does not.

If this system is subject to multiplicative noise that acts linearly on x and y, a variety of terms can exist depending on the structure of the noise. In general we write

$$\dot{A} = pA + q\bar{A} + \Xi(t)(\alpha A + \beta \bar{A}), \tag{2}$$

where $\Xi(t)$ is the noise term. We refer to a "scalar noise" when a single random process is involved and the noise term is written as $\Xi(t)=\xi(t)e^{i\phi}$, where ξ (and from now on) is a real Gaussian white noise with $\langle \xi(t)\xi(t')\rangle = 2\delta(t-t')$. Equation (2) is understood in the sense of Stratonovich [19]. We refer to "vectorial noise" when more than one random process is present. In this work we concentrate on vectorial noise terms of the form $\Xi(t)=\xi_1(t)+i\xi_2(t)$, where the two real processes ξ_1 and ξ_2 are white and Gaussian and are mutually uncorrelated.

Up to a rescaling of time, we can tune the noise amplitude α or β to unity. This amounts in rescaling p and q. We focus on two limiting cases as follows:

(i) $|\alpha|=1$, $\beta=0$ for which the noise does not break the continuous symmetry and p and q are rescaled to $p/|\alpha|^2$ and $q/|\alpha|^2$,

(ii) $\alpha=0$, $|\beta|=1$ for which the noise breaks the symmetry and p and q are rescaled to $p/|\beta|^2$ and $q/|\beta|^2$.

III. ANALYTICAL PREDICTION ON THE ONSET OF INSTABILITY AND ON-OFF INTERMITTENCY

A. Single mode bifurcation

It is instructive to review some of the known results for a system made of one single mode close to its onset. We consider that the nonlinear term saturates the growth and we write

$$\dot{x} = \mu x + \xi x - Cx^m, \tag{3}$$

with C a positive constant and m an integer larger than 2. The Fokker-Plank (FP) equation for the probability density function $P(t,x)$ associated with Eq. (3) is given by

$$\partial_t P = -\partial_x([\mu x - Cx^m]P) + \partial_x(x\partial_x(xP)). \tag{4}$$

Assuming stationarity and integrating we obtain the solution

$$P(x) = \frac{1}{N} x^{\mu-1} e^{-(Cx^{m-1}/m-1)}, \tag{5}$$

where N is a normalization constant. If μ is negative the probability density function (p.d.f.) has a nonintegrable singularity at $x=0$. The only stationary distribution is then $P(x)=\delta(x)$. For positive values of μ the p.d.f. is normalizable and Eq. (5) gives the stationary distribution of x. For $\mu \leq 0$, all initial conditions are ultimately attracted toward zero. For $\mu > 0$, this is not the case: trajectories will not tend to zero but will lead to the stationary distribution (5). Therefore $\mu=0$ defines the onset of instability. We note here that the behavior of $P(x)$ for small values of x is independent of the choice of the nonlinear term.

For $0<\mu<1$ the probability density function has an integrable singularity at the origin, which implies that the trajectory will come arbitrarily close to the origin. This behavior traces back to the on-off intermittency. Indeed, p.d.f. that diverges at 0 and very long durations spent by the trajectories close to $x=0$ are characteristic properties of on-off intermittency [9–14,18,20]. These two properties are controlled by the linear part of the equation since they are related to the evolution when x is very small. In this work we will characterize a regime as on-off intermittent if the resulting p.d.f. diverges at $x=0$. This regime thus corresponds to a power law with exponent between -1 and 0.

Since the criterions for instability and on-off intermittency depend only on the linear terms, it is tempting to derive it ignoring the nonlinearities. For sufficiently small x, we can write the FP equation for the stationary solution as

$$0 = -\partial_x(\mu x P) + \partial_x(x\partial_x(xP)). \tag{6}$$

Since only the linear part of Eq. (3) is considered, Eq. (6) is invariant under transformations $x \to hx$ (for any $h \in R$) and we can look for solutions of the form $P=Cx^{\lambda-1}$. As before if $\lambda \leq 0$ the solution is not integrable for small x. The onset of instability therefore corresponds to $\lambda=0$ when the power-law exponent of the pdf transitions from negative λ to positive λ. Substituting this form of solutions in Eq. (6) we obtain an equation for λ

$$0 = \lambda\mu - \lambda^2 \tag{7}$$

that has two solutions $\lambda=0$ and $\lambda=\mu$. An additional constraint can be obtained from the flux of probability $F(x) \equiv \mu x P - x\partial_x xP = (\mu-\lambda)x^\lambda$ that has to be zero for a stationary solution. This allows us to distinguish between the two solutions. Only the second one (with $\lambda=\mu$) has zero flux, while the $\lambda=0$ solution has in general finite flux. This additional constraint allows calculating the onset of instability defined by the $\lambda=0$ solution with zero flux, which leads to $\mu=0$, as obtained from the solution of the full nonlinear problem.

Finally we can also determine the on-off stability boundary by setting $\lambda=1$ that leads to $\mu=1$.

We now apply this procedure to two-dimensional bifurcations for which analytical calculations of the full pdf cannot, in general, be performed. Results have been obtained for some specific systems. For example, in the case of a Duffing oscillator subject to multiplicative noise, a study of the FP equation allows to estimate the distribution of the oscillator energy. The condition that this distribution is normalizable also predicts the onset of instability [21].

B. Two-dimensional bifurcation with scalar noise

We consider a two-dimensional system

$$\dot{x} = \mu x + \xi(ax + by) + \mathrm{NL}_x,$$
$$\dot{y} = \sigma y + \xi(cx + dy) + \mathrm{NL}_y. \tag{8}$$

NL_i stand for nonlinear terms that we are going to leave unspecified at this point.

Equation (8) describes the motion of a particle in the x-y plane that moves due to a deterministic flow given by $u_x = \mu x + \mathrm{NL}_x$ and $u_y = \sigma y + \mathrm{NL}_y$ plus random displacements described by the noise term. It will be convenient for the following derivations to express the equations in polar coordinates:

$$\dot{r} = r f_r(\theta) + \xi r g_r(\theta) + \mathrm{NL}_r,$$
$$\dot{\theta} = f_\theta(\theta) + \xi g_\theta(\theta) + \mathrm{NL}_\theta. \tag{9}$$

$f_r, f_\theta, g_r, g_\theta$ are functions of θ that depend on the parameters a, b, c, d and μ, σ, namely,

$$f_r = \mu \cos^2(\theta) + \sigma \sin^2(\theta),$$
$$g_r = a \cos^2(\theta) + (b + c)\cos(\theta)\sin(\theta) + d \sin^2(\theta),$$
$$f_\theta = (\sigma - \mu)\cos(\theta)\sin(\theta),$$
$$g_\theta = c \cos^2(\theta) + (d - a)\cos(\theta)\sin(\theta) - b \sin^2(\theta). \tag{10}$$

The Fokker-Planck equation in these coordinates is then written as

$$\partial_t P = -\partial_r(r f_r P) - \partial_\theta(f_\theta P) + \partial_r(r g_r \partial_r(r g_r P)) + \partial_r(r g_r \partial_\theta(g_\theta P)) + \partial_\theta(g_\theta \partial_r(r g_r P)) + \partial_\theta(g_\theta \partial_\theta(g_\theta P)), \tag{11}$$

where P is the probability density function in the polar plane with $\int P dr d\theta = 1$. We have again focused on small values of r and neglected the nonlinear terms. In this limit, the Eq. (11) is invariant under dilatation $r \to hr$. We thus search for stationary solutions of the form

$$P(r,\theta) = r^{\lambda-1}\Pi_\lambda(\theta). \tag{12}$$

Substituting 12 in Eq. (11) leads to the generalized eigenvalue problem,

$$0 = -\lambda f_r \Pi_\lambda - \partial_\theta(f_\theta \Pi_\lambda) + \lambda^2 g_r^2 \Pi_\lambda + \lambda g_r \partial_\theta(g_\theta \Pi_\lambda) + \lambda \partial_\theta[g_\theta g_r \Pi_\lambda] + \partial_\theta[g_\theta \partial_\theta(g_\theta \Pi_\lambda)], \tag{13}$$

where the power law λ is the generalized eigenvalue.

In this work we are interested in finding the marginal value of the control parameters for which a solution other than $P(r,\theta) = \delta(r)$ exists. As in the one-dimensional (1D) case this will be given by setting $\lambda = 0$ in Eq. (13) from which we obtain

$$0 = -\partial_\theta(f_\theta \Pi_0) + \partial_\theta[g_\theta \partial_\theta(g_\theta \Pi_0)]. \tag{14}$$

The solution of which is given by

$$\Pi_0(\theta) = \frac{N}{g_\theta}\exp\left[\int_0^\theta \frac{f_\theta(\theta')}{g_\theta^2(\theta')}d\theta'\right], \tag{15}$$

where the fact that the bifurcation is stationary has been used. Indeed, for a Hopf bifurcation an additional term in Eq. (15) would be present that would express the presence of a current in the θ direction. This is not the case for the systems considered in the following. We now impose that the solution has no radial flux $F(R)$ defined by

$$F(R) \equiv \int_0^R \int_0^{2\pi} \partial_t P dr d\theta = R\int_0^{2\pi} [-f_r P + g_r \partial_r(r g_r P)|_{r=R} + g_r \partial_\theta(g_\theta P)]d\theta. \tag{16}$$

At onset, for $\lambda = 0$, this leads to

$$0 = \int_0^{2\pi} [g_r \partial_\theta(g_\theta \Pi_0) - f_r \Pi_0]d\theta. \tag{17}$$

Equation (17) gives a relation between the control parameters that determines when the system is marginally unstable. A different derivation is obtained based on an asymptotic expansion of Eq. (13). It is presented in the Appendix.

This is our main analytical prediction: the onset of instability is determined by the set of parameters that verify Eq. (17).

In Secs. IV and V we examine specific examples of stationary bifurcations and determine the marginal stability curve based on the flux condition, given by Eq. (17). In addition we determine the domain on which the system displays on-off intermittency by solving numerically the eigenvalue problem [Eq. (13)] using a shooting method and looking for parameters that satisfy $\lambda = 1$.

C. Generalization to vectorial noise

The result is easily generalized when Eq. (8) contains several noise terms ξ_i that are white, Gaussian and independent. We consider the general case

$$\dot{x} = \mu x + \sum_i \xi_i(a_i x + b_i y) + \mathrm{NL}_x,$$
$$\dot{y} = \sigma y + \sum_i \xi_i(c_i x + d_i y) + \mathrm{NL}_y. \tag{18}$$

Then, Eq. (9) becomes

$$\dot{r} = r f_r(\theta) + r\sum\nolimits_i \xi_i g_{i_r}(\theta) + \mathrm{NL}_r,$$

$$\dot{\theta} = f_\theta(\theta) + \sum_i \xi_i g_{i_\theta}(\theta) + \mathrm{NL}_\theta, \tag{19}$$

where g_{i_r} and g_{i_θ} have the form given in Eq. (10). In the FP Eq. (11), diffusive terms associated to each noise appear. The equation for the stationary angular distribution at onset is then

$$0 = -\partial_\theta(f_\theta \Pi_0) + \sum_i \partial_\theta[g_{i_\theta}\partial_\theta(g_{i_\theta}\Pi_0)], \tag{20}$$

and the criterion for marginal stability reads

$$0 = \int_0^{2\pi}\left[\sum_i g_r \partial_\theta(g_{i_\theta}\Pi_0) - f_r \Pi_0\right] d\theta. \tag{21}$$

IV. CASE (i): NOISE THAT DOES NOT BREAK CONTINUOUS SYMMETRY

A. Scalar noise

We begin by investigating a system for which the noise term does not break rotational symmetry $A \to e^{i\psi}A$, and write it

$$\dot{A} = pA + q\bar{A} + e^{i\phi}A\xi + \mathrm{NL}. \tag{22}$$

Without loss of generality, we restrict to $0 \le \phi \le \pi/2$. In terms of the original variables (x,y) the system is written as

$$\dot{x} = \mu x + \xi[\cos(\phi)x - \sin(\phi)y] + \mathrm{NL}_x,$$

$$\dot{y} = \sigma y + \xi[\sin(\phi)x + \cos(\phi)y] + \mathrm{NL}_y. \tag{23}$$

In the absence of noise a particle initially placed at $x = x_0$, $y = y_0$ would move along the trajectory $(x/x_0) = (y/y_0)^{\mu/\sigma}$. We refer to these trajectories as deterministic flow lines. Similarly, in the absence of the deterministic part of Eq. (23) ($\mu = \sigma = 0$) particles are also restricted to move along specific lines. For $\phi = \pi/2$ the noise drives motion along the circles $x^2 + y^2 = x_0^2 + y_0^2$; for $\phi = 0$ it acts along the lines $x/x_0 = y/y_0$, and for all other values it acts on spiral trajectories that can be written in polar coordinates as $r/r_0 = \exp[\tan(\phi)(\theta - \theta_0)]$, see Fig. 2. We refer to these lines as diffusion lines since the probability density is diffused along these lines. Since the flow lines and the diffusion lines in general intersect each other, the trajectory of a particle is not restricted to either the diffusion or flow lines.

The pdf for the marginal case Π_0 obtained from Eq. (15) is

$$\Pi_0(\theta) = \frac{1}{N}\exp\left[\frac{(\sigma - \mu)\cos(2\theta)}{\sin(\phi)}\right]. \tag{24}$$

Apart from the singular case $\phi = 0$ that we discuss separately, the stability of the system can be determined by the flux criterion [Eq. (17)]. The resulting stability boundaries for the case $\phi = \pi/2$ and $\phi = \pi/8$ are shown in Figs. 3 and 4, respectively, with dark solid line. In addition to the stability boundaries, the boundaries of on-off intermittency are calculated numerically from the eigenvalue problem [Eq. (13)], and are shown in Figs. 3 and 4 with a dashed line. Qualitatively

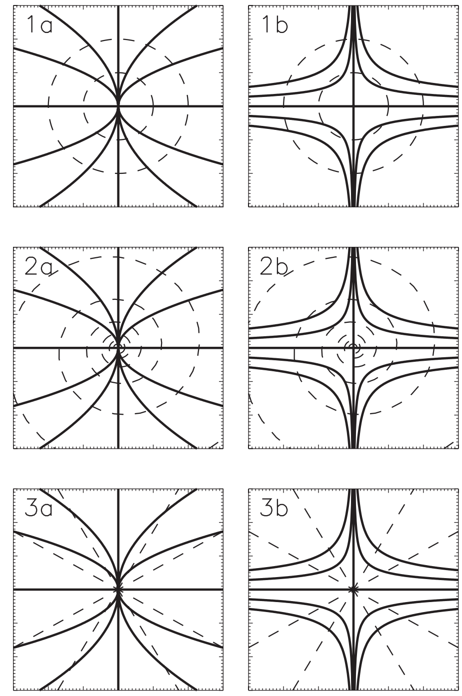

FIG. 2. Deterministic flow lines (solid) and diffusion lines (dashed) for Eq. (22) in the x-y plane. Panels "a" refer to cases that $\mu/\sigma > 0$ and panels "b" refer to cases for which $\mu/\sigma < 0$. $\phi = \pi/2$ for panels 1, $\phi = \pi/4$ for panels 2, and $\phi = 0$ for panels 3.

similar results are obtained for all ϕ such that $0 < \phi \le \pi/2$.

The Langevin (L-) Eqs. (23) are also solved numerically using three different models of nonlinearities. For the first model, the nonlinear terms $\mathrm{NL}_x = -x(x^2 + y^2)$ and $\mathrm{NL}_y = -y(x^2 + y^2)$ were used. They act only on the r direction and preserve the original symmetry. For the second model, the more general nonlinear terms $\mathrm{NL}_x = -(x + \omega y)(x^2/a + y^2/b)$ and $\mathrm{NL}_y = -(y - \omega x)(x^2/a + y^2/b)$ were used that also introduce a θ dependence of the nonlinearity. Finally, the nonlinear terms were modeled using a no-flux boundary condition at $|x| = 1$ and $|y| = 1$. For all cases the stability boundaries and the on-off boundaries were verified. For the $\phi = \pi/2$ case with $\mu = 1.1$ and $\sigma = -2$, Fig. 5 shows the phase space of the solution where we have plotted 10^4 samples of the numerical solution of the L equation obtained at different times. Simi-

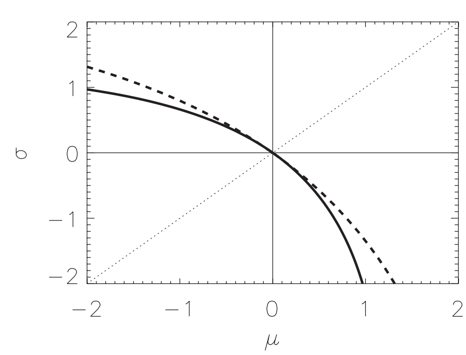

FIG. 3. Stability (solid line) and on-off boundaries (dashed lines) for the case of scalar noise [Eq. (22)] that does not break symmetry with $\phi = \pi/2$.

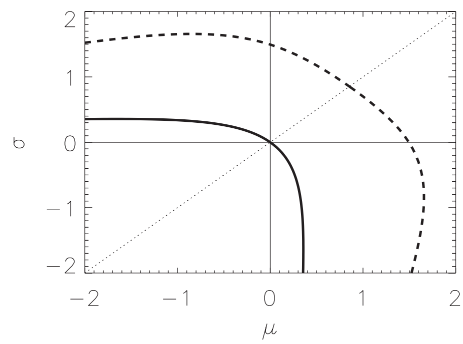

FIG. 4. Stability (solid line) and on-off boundaries (dashed line) for the case of scalar noise [Eq. (22)] that does not break symmetry with $\phi=\pi/8$.

larly, Fig. 6 displays the $\phi=\pi/2$ case with $\mu=1$ and $\sigma=0$. For these values of the parameters μ and σ, the solution displays on-off intermittency as indicated by the high density close to $x=y=0$. Similar figures are obtained for other parameter values in the range located between the two boundaries shown in Figs. 3 and 4.

Equation (15) is singular for the $\phi=0$ case. This singular behavior occurs because the flow and the diffusion lines are aligned along the x and the y axes (see Fig. 3, panels a and b). Due to this alignment if a particle is initially placed on one of these axes it will remain on this axis. As ϕ tends to zero, Π_0 obtained from Eq. (24) becomes more peaked along the most unstable axis and tends to a delta function. In the limit $\phi\to 0$ the stationary p.d.f. is $\Pi_0(\theta)=N_1\delta(\theta)+N_2\delta(\theta+\pi)$, (if $\mu>\sigma$) and $\Pi_0(\theta)=N_1\delta(\theta+\pi/2)+N_2\delta(\theta-\pi/2)$ (if $\mu<\sigma$), where N_1 and N_2 are normalization constants. For random initial conditions the criterion for instability [Eq. (17)] reads $\max[\mu,\sigma]>0$. In addition, based on the same assumption, the on-off behavior is derived from that of the 1D system $\max[\mu,\sigma]<1$. The parameter space is shown in Fig. 7.

In conclusion, in the presence of a scalar noise that does not break the continuous symmetry, the onset of instability is postponed or is equal to the deterministic onset. Both onsets are equal when $\mu=\sigma$ or in the singular case $\phi=0$. This means that when the system displays a continuous symmetry, a scalar noise that does not break this symmetry has no effect

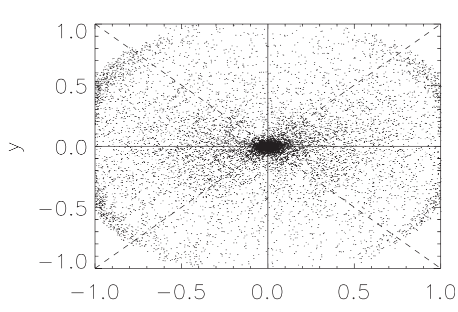

FIG. 5. Location in phase space of 10^4 samples of the numerical solution of the Langevin equation obtained at different times for a scalar noise that does not break the continuous symmetry [Eq. (22)] with $\mu=-2$, $\sigma=1.1$ and $\phi=\pi/2$. No flux boundaries at $|x|=|y|=1$ were used.

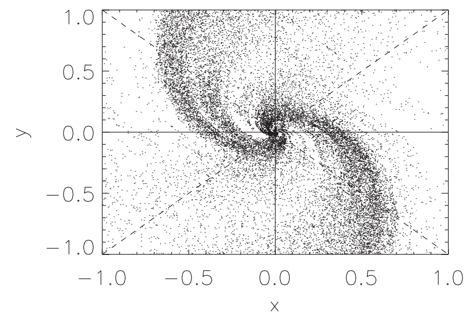

FIG. 6. Location in phase space of 10^4 samples of the numerical solution of the Langevin equation obtained at different times for a scalar noise that does not break the continuous symmetry [Eq. (22)] with $\mu=0$, $\sigma=1$ and $\phi=\pi/8$. No flux boundaries at $|x|=|y|=1$ were used.

on the onset of instability. Concerning the regime of on-off intermittency, we point out that for $\phi=\pi/2$, i.e., a noise that rotates the phase of A, on-off intermittency occurs in a narrow set of parameters. It is even suppressed when the system displays the continuous symmetry. This is a consequence of the noise acting along θ, leaving the dynamics along r to be independent of the noise.

B. Vectorial noise

We now investigate the case of a vectorial noise that does not break the continuous symmetry and consider

$$\dot{A} = pA + q\bar{A} + [\xi_1(t) + i\xi_2(t)]A + \mathrm{NL}. \tag{25}$$

This noise is the sum of two scalar noise terms with $\phi=0$ and $\phi=\pi/2$ so that their diffusion lines intersect perpendicularly at each point. The eigenvalue equation for λ in this case simplifies to

$$0 = -\lambda f_r \Pi - \partial_\theta(f_\theta \Pi) + \lambda^2 \Pi + \partial_\theta^2 \Pi. \tag{26}$$

The stationary angular distribution at onset is easily obtained:

$$\Pi_0(\theta) = e^{q\cos(2\theta)/2}/N, \tag{27}$$

from which the marginal stability curve is expressed as

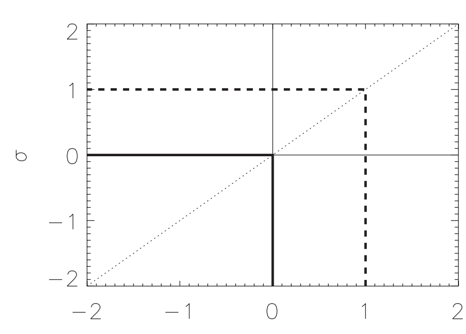

FIG. 7. Stability (solid line) and on-off intermittency boundaries (dashed line) for the scalar noise that does not break the continuous symmetry [Eq. (22)] with $\phi=0$.

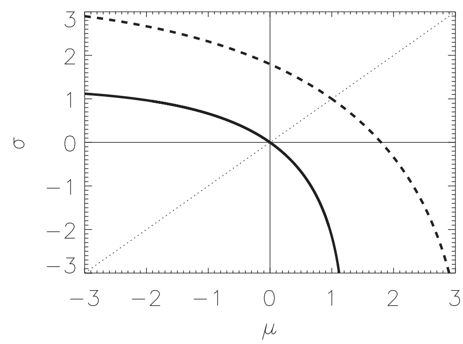

FIG. 8. Stability (solid line) and on-off intermittency boundaries (dashed line) for the vectorial noise that does not break symmetry [Eq. (25)].

$$\int_{\theta=0}^{2\pi}[p+q\cos(2\theta)]e^{q\cos(2\theta)/2}d\theta=0. \quad (28)$$

In this case, the result can be written in a closed analytical form, namely,

$$p=\omega\frac{J_1(\omega)}{J_0(\omega)}, \quad (29)$$

with $\omega=i\frac{q}{2}$, and J_n is the Bessel function of the first kind of order n. The onset of instability, together with the numerically calculated boundary of on-off intermittency, is presented in Fig. 8. The location in phase space of 10^4 samples of the numerical solution of the Langevin equation is shown in Fig. 9.

Results are qualitatively similar to those of the scalar noise with $0<\phi<\pi/2$. Namely, the onset in the presence of noise is larger than the deterministic one. They are equal only when the system displays the continuous symmetry.

V. CASE (ii): NOISE THAT BREAKS THE CONTINUOUS SYMMETRY

A. Scalar noise

In this section we investigate a system for which the noise term breaks the rotational symmetry $A\to e^{i\psi}A$. More specifically we consider

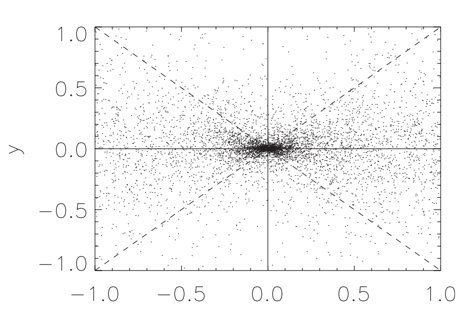

FIG. 9. Location in phase space of 10^4 samples of the numerical solution of the Langevin equation obtained at different times for a vectorial noise that does not break the continuous symmetry [Eq. (25)] with $\mu=0$, $\sigma=1$ and $\phi=\pi/4$. No flux boundaries at $|x|=|y|=1$ were used.

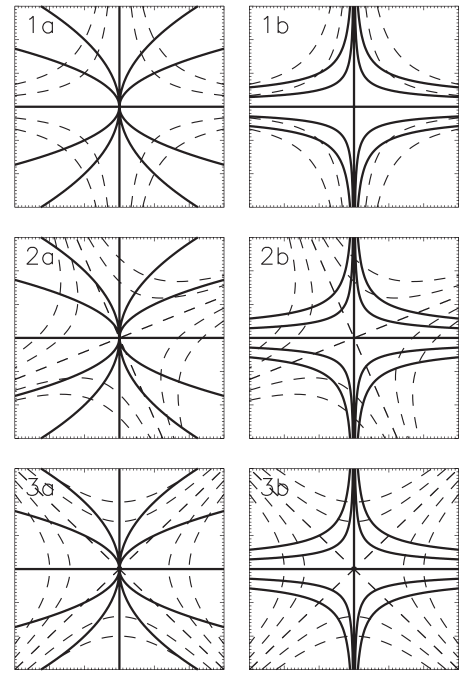

FIG. 10. Deterministic flow lines (solid) and diffusion lines (dashed) for Eq. (30) in the x-y plane. Panels "a" refer to cases with $\mu\sigma>0$ and panels "b" refer to cases with $\mu\sigma<0$. $\phi=0$ for panels 1, $\phi=\pi/4$ for panels 2, and $\phi=\pi/2$ for panels 3.

$$\dot{A}=pA+q\bar{A}+e^{i\phi}\bar{A}\xi+\mathrm{NL}. \quad (30)$$

In this case, the diffusion lines are given by the hyperbolas $\sin(\phi)(x^2-y^2)-2\cos(\phi)xy=c$, whose asymptote axes are given by $y=\pm\tan(\phi/2)x$. For $\phi=0$ the two axis of the hyperbolas coincide with the x and y axis, while for $\phi=\pi/2$ the two axis of the hyperbolas coincide with the two diagonals $x=\pm y$ (see Fig. 10).

In terms of the variables (x,y) the system is written as

$$\dot{x}=\mu x+\xi[\cos(\phi)x+\sin(\phi)y]+\mathrm{NL}_x,$$

$$\dot{y}=\sigma y+\xi[\sin(\phi)x-\cos(\phi)y]+\mathrm{NL}_y, \quad (31)$$

and the solution Π_0 of 15 is

$$\Pi_0(\theta)=\frac{[\tan(\theta-\phi/2)]^{[(\sigma-\mu)/4]\cos(\phi)}}{N\sin(\phi-2\theta)}\exp\left[\frac{(\mu-\sigma)\sin(\phi)}{4\sin(2\theta-\phi)}\right]. \quad (32)$$

The axis of the hyperbolas split the phase space into four "quarters." Close to the two axes of the hyperbolas ($\theta=\phi/2+\delta\theta$ with $\delta\theta\ll1$), Π_0 behaves like $\Pi_0(\theta)\sim\exp[(\mu-\sigma)\sin(\phi)/4\delta\theta]/(\delta\theta)$. If $\mu>\sigma$ then for values of θ slightly larger than $\phi/2$, Π_0 has a smooth behavior, and for slightly smaller than $\phi/2$, Π_0 has singular behavior. It is the opposite if $\mu<\sigma$. For Π_0 of the form of Eq. (32) to be normalizable, the following restrictions need to be made: if $\mu>\sigma$ ($\mu<\sigma$), Π_0 is obtained from Eq. (32) within the angles $(\phi/2,-\phi/2)$ and $(\pi-\phi/2,3\pi/2-\phi/2)$ $[(\phi/2,\phi/2+\pi/2)$ and $(\pi+\phi/2,3\pi/2+\phi/2)]$, and is zero otherwise. This dependence on θ is easily understood: on the asymptote of the hyperbo-

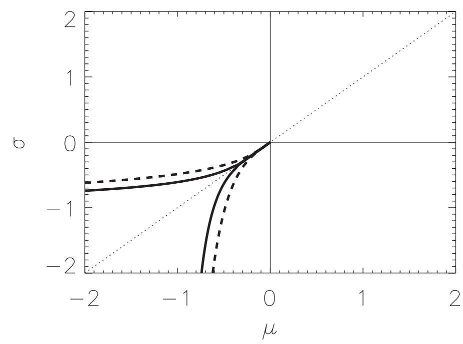

FIG. 11. Stability (solid line) and on-off intermittency boundaries (dashed line) for the scalar noise that breaks symmetry [Eq. (30)] with $\phi=\pi/2$.

las, the noise acts only radially and the evolution in the θ direction is controlled only by the deterministic flow. As displayed in Fig. 10, there is a sign determined flux of probability toward the quarters that contain the most unstable axis. The stationary distribution thus vanishes in the two other quarters. With these restrictions on the allowed angles θ, the stability of the system can be determined by the flux criterion [Eq. (17)].

The resulting stability boundaries for the case $\phi=\pi/2$ and $\phi=\pi/20$ are shown in Figs. 11 and 12, respectively, with dark solid line. The boundaries of on-off intermittency that were calculated by numerically solving the eigenvalue problem [Eq. (13)], are shown with dashed lines. Qualitatively similar results are obtained for all angles $0<\phi\leq\pi/2$. Figure 13 shows the location in phase space of 10^4 samples of the numerical solution of the Langevin equation obtained at different times for the $\phi=\pi/2$ case with $\mu=1.1$ and $\sigma=-2$. Similarly Fig. 14 shows the case with $\phi=\pi/2$, $\mu=1$, and $\sigma=0$.

The $\phi=0$ case is singular because the modes are not coupled at linear order. Π_0 has a singular structure along the axis $x=0$ and $y=0$ on which the deterministic flow lines are aligned with the diffusion lines. Since there is no flux across the $x=0$, $y=0$ axis with no loss of generality we can consider only the $x>0$ and $y>0$ quarter, and for simplicity let us assume that the nonlinearities are modeled by zero flux boundary conditions at $|x|=|y|=1$. In Fig. 15 we show in log-log scale the diffusion lines for this specific noise. If $\mu+\sigma>0$ the deterministic part of the flow will drive the particle closer to the point $x=y=1$. If $\mu+\sigma<0$ but one of the growth rates is positive (say σ) the particle will be attracted to the unstable axis and the system will be reduced to a one-dimensional problem. If both modes have negative growth rates, the particle will move toward $x=y=0$. The onset of instability is then given by $\max[\mu,\sigma]=0$. The stability and on-off intermittency boundaries based on this description are shown in Fig. 16 and we have verified these results numerically for the no-flux boundary conditions. However, the on-off intermittency turns out to depend on the choice of nonlinear terms. This is due to the singular structure of the $\phi=0$ case.

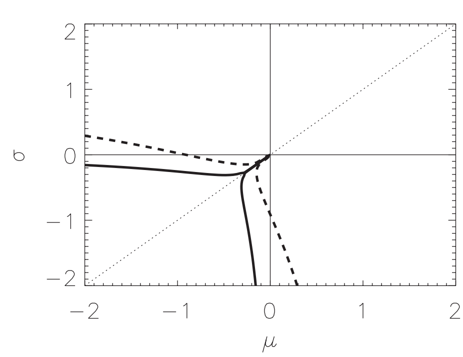

FIG. 12. Stability (solid line) and on-off intermittency boundaries (dashed line) for the scalar noise that breaks symmetry [Eq. (30)] with $\phi=\pi/20$.

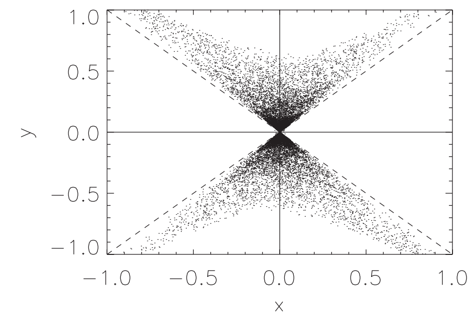

FIG. 13. Location in phase space of 10^4 samples of the numerical solution of the Langevin equation obtained at different times for a scalar noise that breaks the continuous symmetry [Eq. (30)] with $\mu=-2$, $\sigma=-0.72$ and $\phi=\pi/2$. No flux boundaries at $|x|=|y|=1$ were used.

In general, in the presence of noise, the onset of instability is shifted to smaller values of the control parameters compared to the deterministic case. The onsets are equal when $\mu=\sigma$ or in the singular case $\phi=0$. For a given intensity of a scalar noise that breaks the continuous symmetry, the solution $x=y=0$ is more stable when the system displays the continuous symmetry. The effect is quite strong: even a very small breaking of the continuous symmetry is enough to generate a large change in the value of the onset. Expansion of the equation for the marginal stability curve close to $\mu=\sigma=0$, leads to $|\mu-\sigma|\sim e^{2/\sigma}$, i.e., $\mu\sim 2/log(|\mu-\sigma|)$. For instance, for a broken symmetry of $\mu-\sigma=10^{-4}$, the onset is shifted to $\mu=-0.22$. We also point out that on-off intermit-

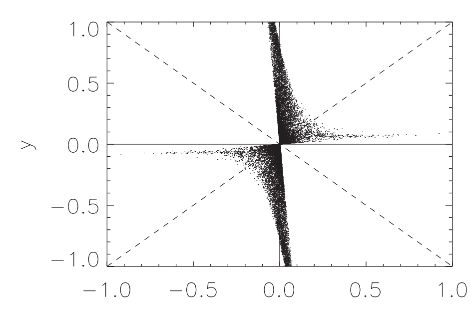

FIG. 14. Location in phase space of 10^4 samples of the numerical solution of the Langevin equation obtained at different times for a scalar noise that breaks the continuous symmetry [Eq. (30)] with $\mu=-2$, $\sigma=0$ and $\phi=\pi/20$. No flux boundaries at $|x|=|y|=1$ were used.

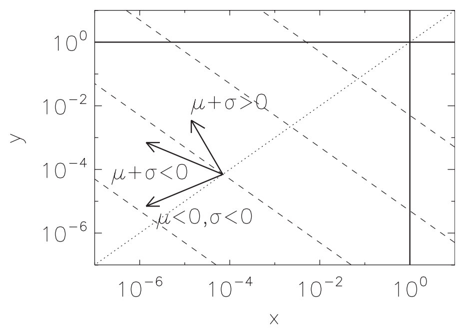

FIG. 15. Diffusion lines in log-log scale for the noise term that breaks symmetry [Eq. (30)] with $\phi=0$.

tency disappears for all values of ϕ, when the system displays the continuous symmetry.

B. Vectorial noise

We now investigate the case of a vectorial noise that breaks the continuous symmetry and consider

$$\dot{A} = pA + q\bar{A} + [\xi_1(t) + i\xi_2(t)]\bar{A} + \mathrm{NL}. \tag{33}$$

The eigenvalue equation simplifies to

$$0 = -\lambda f_r \Pi - \partial_\theta(f_\theta \Pi) + \lambda^2\Pi + 2\lambda\Pi + \partial_\theta^2\Pi. \tag{34}$$

The stationary angular distribution is identical to Eq. (27) obtained in the case of vectorial noise that does not break symmetry. The marginal stability curve is then given by

$$\int_{\theta=0}^{2\pi} [p + 2 + q\cos(2\theta)]e^{q\cos(2\theta)/2}d\theta = 0. \tag{35}$$

Here also a closed analytical expression can be obtained

$$p = -2 + \omega\frac{J_1(\omega)}{J_0(\omega)}, \tag{36}$$

with $\omega = i\frac{q}{2}$. The onset of instability, together with the numerically calculated boundary of on-off intermittency, is presented in Fig. 17.

Results are qualitatively different from the case of a scalar noise. The onset of instability is shifted to smaller values of

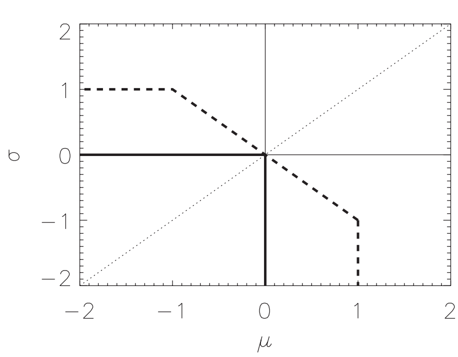

FIG. 16. Stability (solid line) and on-off intermittency boundaries (dashed line) for the scalar noise that breaks symmetry [Eq. (30)] with $\phi=0$.

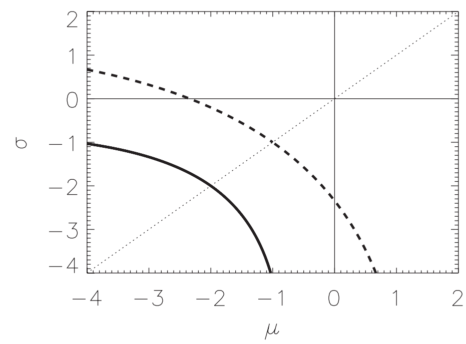

FIG. 17. Stability (solid line) and on-off intermittency boundaries (dashed line) for the vectorial noise that breaks symmetry [Eq. (33)].

the control parameter and on-off intermittency occurs above onset even when the system displays the continuous symmetry.

VI. CONCLUSION

We have developed an analytical approach that allows calculating the onset of instability and the domain of on-off intermittency for a planar system subject to multiplicative noise.

These results are obtained from the Fokker-Plank equation of the system and only depend on the linear terms in the Langevin equation. At first sight, it may seem surprising. Indeed, it is well known that when multiplicative noise is considered, nonlinearities must be taken into account, otherwise results are likely to be biased by rare events where the noise drives huge values of the unstable mode. This is for instance the case when stability is determined based on the growth of different moments calculated from the linear part of the Langevin equation. However our results implicitly require the presence of nonlinear terms when the existence of a stationary solution is assumed. Our predictions for the onset of instability are therefore independent of the nonlinear terms as long as a stationary distribution exists. This is also the case for the on-off boundaries, apart from the singular case where diffusion and flow lines are aligned.

We have calculated the effect of several kinds of noise terms on an instability in a system that displays or has weakly broken a continuous symmetry. What is observed is that noise that does not break the continuous symmetry postpones the onset when the symmetry is broken ($\mu \neq \sigma$). It does not modify the onset in the continuous symmetric case. A scalar noise that breaks the continuous symmetry anticipates the onset if the symmetry is weakly broken. Even a small symmetry breaking drastically reduces the onset: the system turns out to be more unstable when one of the modes is made more stable. Results are different for a vectorial noise that breaks the continuous symmetry: the onset of instability is anticipated even in the continuous symmetric case.

In general on-off intermittency takes place above onset. It disappears in a few cases: when the system displays the continuous symmetry for a scalar noise that breaks the symmetry and for a scalar noise that does not break the symmetry but acts only in the angular direction (case $\phi = \pi/2$).

In the dynamo context, we point out that fluctuations that break the continuous symmetry lower the onset. For the case of a scalar noise, when one mode is more damped, the system turns out to be more unstable. This is a regime both surprising and of possible experimental interest. The determination of the amplitude of these effects depending on the spatial structures of the velocity fluctuations remains to be done.

In this paper we considered only the case of stationary bifurcations. Following work could study how these results are modified for a Hopf bifurcation. Extension of our analytical approach to bifurcations in higher dimensions could also be considered.

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF THE FLUX CONDITION

In this appendix, we derive the stability (flux) condition based on an asymptotic expansion. It is presented for the case of scalar noise but it is easily generalized to the vectorial noise case. Let μ_0, σ_0 be the value of the control parameters for which the system is marginally stable. We write the stationary p.d.f. as $P = r^{\lambda-1}\Pi(\theta)$. At onset (μ_0, σ_0) the solution of the FP equation transitions from a non-normalizable form $\lambda < 0$ to a normalizable form $\lambda > 0$. Close to onset we expand Eq. (13) in powers of λ and write $\Pi = \Pi_0 + \lambda \Pi_1 + \ldots$, $\mu = \mu_0 + \lambda \mu_1$, and $\sigma = \sigma_0 + \lambda \sigma_1$. The eigenvalue Eq. (13) for λ can be written as

$$L_{\mu,\sigma}\Pi + \lambda M_{\mu,\sigma}\Pi + \lambda^2 N_{\mu,\sigma}\Pi = 0, \tag{A1}$$

where the operators $L_{\mu,\sigma}, M_{\mu,\sigma}, N_{\mu,\sigma}$ are given by

$$L_{\mu,\sigma}\Pi = \partial_\theta[g_\theta \partial_\theta (g_\theta \Pi)] - \partial_\theta[f_\theta \Pi], \tag{A2}$$

$$M_{\mu,\sigma}\Pi = \partial_\theta[g_\theta g_r \Pi] + g_r \partial_\theta[g_\theta \Pi] - f_r \Pi, \tag{A3}$$

$$N_{\mu,\sigma}\Pi = g_r^2 \Pi. \tag{A4}$$

At zeroth order in power of λ we get

$$L_{\mu_0,\sigma_0}\Pi_0 = 0,$$

whose solution is given by Eq. (15). At next order we have

$$L_{\mu_0,\sigma_0}\Pi_1 + M_{\mu_0,\sigma_0}\Pi_0 + \mu_1 [\partial_\mu L_\mu]_{\mu=\mu_0}\Pi_0 = 0. \tag{A5}$$

Averaging Eq. (A5) with respect to θ we obtain the solvability condition

$$\int M_{\mu_0,\sigma_0}\Pi_0 d\theta = 0, \tag{A6}$$

where we have used the property of the $L_{\mu,\sigma}$ operator $\int L_{\mu,\sigma}\Pi d\theta = 0$. Substituting the expression for $M_{\mu,\sigma}$ we obtain the flux condition [Eq. (17)].

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